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FIRST SEMESTER EXAMINATIONS, 2024/2025 SESSION

MATH40[illegible]: Differential Equations.

INSTRUCTIONS: Answer ANY FOUR Questions. Time allowed: $2\frac{1}{2}$ hours

1. a) Obtain the series solution of the differential equations: $y'' - 2xy' + 4y = x^2 + x,\ y(0)=1,\ y'(0)=-2$ about the ordinary point $x=0$.

   b) Express in terms of gamma and beta functions the integral $\int_0^1 x^m (1-x^n)^p\, dx$. Hence evaluate the integral $\int_0^1 x^3 (1-x^2)^5\, dx$

2. a) Solve the Sturm-Liouville equation $(xy')' + \frac{\lambda}{x} y = 0,\ y(1)=0,\ y(e)=0$ to obtain the eigenvalues and the corresponding eigenfunctions for the problem. Show that the set of eigenfunctions forms an orthogonal and orthonormal set.

   b) Show that $4x^3 - 3x^2 + 2x + 1 = \frac{8}{5}P_3(x) - 2P_2(x) + \frac{22}{5}P_1(x)$, where $P_1(x), P_2(x), P_3(x)$ are Legendre polynomials.

3. Define a regular and an irregular singular point for the differential equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$.

   Hence show that the point $x=0$, is a regular singular point of the differential equation, $2x(1-x)y'' + (5-7x)y' - 3y = 0$. Hence obtain the series solution of the differential equation about the point $x=0$.

4. a) Find the general solution of the Legendre's differential equation $(1-x^2)y'' - 2xy' + p(p+1)y = 0$.

   b) Use the generating function for the Legendre polynomial $(P_n(x))$ to show that $P_{n+1}(x) = \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x)$. If $P_0(x)=1$, $P_1(x)=x$ find $P_2(x)$, $P_3(x)$ and $P_4 x)$

5. a) If $J_k(x)$ is the Bessel function of the first kind, show that :

   (i) $\frac{d}{dx}(J_k(x)) = \frac{1}{2}(J_{k-1}(x) + J_{k+1}(x))$ ii) $J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x)$

   b) If $P_n(x)$ and $P_m(x)$ are the Legendre Polynomials of order n and m then show that

   i) $\int_{-1}^{1} p_n(x) p_m(x) dx = 0$ if $n \neq m$ ii) $\int_{-1}^{1} p_n(x))^2 dx = \frac{2}{2n+1}$

6. a) State the existence and uniqueness theorem for the IVP ; $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ in $R: |x - x_0| \le a \quad |y - y_0| \le b$.

   b) Show that the function $f(x,y) = x^2 + 2xy + y^2$ satisfies the Lipchitz condition on R and find the Lipchitz constant. Hence show that the IVP $\frac{dy}{dx} = x^2 + 2xy + y^2$, $y(1) = 2$, $R: |x-1| \le 2,\ |y-2| \le 2$ has a unique a solution. c) Solve the system of differential. Equations $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$ given that x = 0 and y = 0 when t = 0.

1a. $y'' - 2xy' + 4y = x^2 + x$, $y(0) = 1$, $y'(0) = -2$ about the ordinary point $x = 0$.

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

• Substitute to the main equation we get;

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = x^2 + x$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = x^2 + x$$

• Change at index to $n \longrightarrow n+2$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} x^n + (4 - 2n) a_n x^n\right] = x^2 + x$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (4 - 2n) a_n\right] x^n = x^2 + x$$

Compare coefficients of x^2 and x Since we assume $y(x) = \sum_{n=0}^{\infty} a_n x_n$

$\Rightarrow y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$y(0) = 1 \Rightarrow a_0 = 1$ and $y'(0) = -2 \Rightarrow a_1 = -2$.

$x^2 + x = \sum_{n=0}^{\infty} b_n x^n$, $b_1 = 1$ and $b_2 = 1$, $C_n = 0$ for $n \geq 3$.

$$\Rightarrow (n+2)(n+1) a_{n+2} + (4 - 2n) a_n = 0 \quad \text{for } n \geq 3.$$

$$(n+2)(n+1) a_{n+2} = -a_n (4 - 2n)$$

$$a_{n+2} = \frac{-a_n (4 - 2n)}{(n+2)(n+1)}$$

For $n = 0$ $\quad a_2 = \dfrac{-4a_0}{2\times 1} = \dfrac{0 - 4(a)}{2} =$

$n = 1$ $\quad a_3 = \dfrac{-2a_1}{3\times 2} = \dfrac{1 + 2a_1}{6} =$

$n = 2$ $\quad a_4 = \dfrac{-0.a_2}{4.3} = \dfrac{1}{12}$

$n = 3$ $\quad a_5 = \dfrac{+2(5/6)}{20} = \dfrac{10/6}{20} = \dfrac{1}{12}$

$n = 4$ $\quad a_6 = \dfrac{-(-4)a_4}{6\times 5} = \dfrac{4 \cdot 1/12}{30} = \dfrac{1}{90} \ldots$

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 + x^3 + \dots$

The final solution as given by ;

$$y(x) = 1 - 2x - 2x^2 + \frac{5}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{12}x^5 + \frac{1}{90}x^6 + \dots$$

b.) $\int_0^1 x^m (1-x^n)^p \, dx$.

Let $x^n = u \Rightarrow x = u^{\frac{1}{n}}$ and $dx = \frac{1}{n} u^{\frac{1}{n}-1} du$.

Substituting to the equation

$$\int_0^1 (u^{\frac{1}{n}})^m (1-u)^p \cdot \frac{u^{\frac{1}{n}-1}}{n} du = \frac{1}{n}\int_0^1 u^{\frac{m}{n}+\frac{1}{n}-1}(1-u)^p du$$

$$= \frac{1}{n}\int_0^1 (1-u)^p \cdot u^{(\frac{m}{n}+\frac{1}{n})-1} du$$

This is similarly to $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$

$\Rightarrow b = \frac{m+1}{n}$ and $a = p+1$

$$\Rightarrow B\left(p+1, \frac{m+1}{n}\right) = \frac{\Gamma a \, \Gamma b}{\Gamma a+b} = \frac{\Gamma p+1 \, \Gamma \frac{m+1}{n}}{\Gamma \frac{p+1}{1} + \frac{m+1}{n}} = \frac{\Gamma p+1 \, \Gamma \frac{m+1}{n}}{\Gamma \frac{n(p+1)+m+1}{n}}$$

Hence evaluating $\int_0^1 x^5(1-x^2)^9 dx$.

From the above equation, after comparing, we see that ;

$m = 5$, $n = 2$ and $p = 9$.

$$\Rightarrow \int_0^1 x^5(1-x^2)^9 = \frac{\Gamma 9+1 \, \Gamma \frac{5+1}{2}}{\Gamma \frac{2(9+1)+5+1}{2}} = \frac{\Gamma 10 \, \Gamma 3}{\Gamma \frac{26}{2}} = \frac{\Gamma 10 \, \Gamma 3}{\Gamma 13}$$

$$= \frac{9! \cdot 2!}{12!} = \frac{\cancel{9!} \cdot \cancel{2!}}{12 \times 11 \times \cancel{10}^{5} \times \cancel{9!}} = \frac{1}{12 \times 11 \times 5} = \frac{1}{660}$$

2.

a. $(xy')' + \frac{\lambda}{x}y = 0$, $y(1) = 0$, $y(e) = 0$, expanding the equation, we get ; $xy'' + y' + \frac{\lambda}{x}y = 0$. The multiply through by x.

$$x^2y'' + xy' + \lambda y = 0.$$

We can solve the D.E by assuming $y(x) = x^r$, $\Rightarrow y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substitute result into the equation.

$$x^2 \cdot r(r-1)x^{r-2} + x \cdot rx^{r-1} + \lambda x^r = 0$$

$$x^r\left[r(r-1) + r + \lambda\right] = 0$$

$$r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda} \Rightarrow r = \pm i\sqrt{\lambda}.$$

And solution for complex roots as given by ;

$$y(x) = A\cos(\sqrt{\lambda}\ln x) + B\sin(\sqrt{\lambda}\ln x)$$

Apply the boundary conditions ;

$$y(1) = 0 \Rightarrow A\cos(0) + B\sin(0) = 0$$

$A = 0$. This reduces the solution to

$y(x) = B\sin(\ln x\sqrt{\lambda})$. Then the second condition ;

$$y'(e) = B\frac{\sqrt{\lambda}}{x}\cos(\ln x\sqrt{\lambda}).$$

$$y'(e) = 0 = B\frac{\sqrt{\lambda}}{e}\cos(\ln e\sqrt{\lambda}) = B\frac{\sqrt{\lambda}}{e}\cos\sqrt{\lambda} = 0$$

And since $B \neq 0$ and $\lambda \neq 0$

$\Rightarrow \cos\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = \frac{\pi(2n+1)}{2}$, for $n = 1, 2, 3, \cdots$

$\Rightarrow \lambda_n = \frac{\pi^2(2n+1)^2}{4}$. (Eigen values)

The Eigenfunctions are given by

$$y_n(x) = B\sin(\ln x\sqrt{\lambda})$$

$$= \sin\left(\frac{\pi(2n+1)}{2}\ln x\right)$$

From the equation $(xy')' + \frac{\lambda}{x} y = 0$ we know that the weight function is $\lambda w(x) = \frac{\lambda}{x} \Rightarrow w(x) = \frac{1}{x}$.

And to check orthogonality with the given interval $[1, e]$

$$\int_1^e y_m(x)\, y_n(x)\, w(x)\, dx = 0 \quad \text{for } m = 1,2,3,\dots \text{ and } n = 1,2,3,\dots \text{ and } m \neq n$$

$$\Rightarrow \int_1^e \sin\left[\left(m+\tfrac{1}{2}\right)\pi \ln x\right] \cdot \sin\left[\left(n+\tfrac{1}{2}\right)\pi \ln x\right] \cdot \frac{1}{x}\, dx .$$

Solve by substitution method, let $t = \ln x \Rightarrow x = e^t$

$\Rightarrow dx = e^t dt \Rightarrow \frac{dx}{x} = dt$ [since $x = e^t$]

And the limit will be ;

0 when $x = 1$ and 1 when $x = e$.

So the new integral becomes ;

$$\int_0^1 \sin\left(\left(m+\tfrac{1}{2}\right)\pi t\right) \sin\left(\left(n+\tfrac{1}{2}\right)\pi t\right) dt \quad \text{when } m \neq n \text{ and } \tfrac{1}{2} \text{ when } m = n$$

Which the standard sine orthogonal function.

Hence $y_n(x)$ and $y_m(x)$ are orthogonal on $[1, e]$.

And orthonormal when $m = n$.

b. The Legendre polynomials are given by

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Now plug this to the equation, we obtain ;

$$\frac{8}{5} P_3(x) - 2P_2(x) + \frac{22}{5} P_1(x) = \frac{\cancel{8}^{4}}{5} \cdot \frac{1}{\cancel{2}_{1}}(5x^3 - 3x) - \cancel{2} \cdot \frac{1}{\cancel{2}}(3x^2 - 1) + \frac{22}{5} \cdot x$$

$$= 4x^3 - \frac{12}{5}x - 3x^2 + 1 + \frac{22}{5} = 4x^3 - 3x^2 + \left(\frac{22}{5} - \frac{12}{5}\right)x + 1$$

$$= 4x^3 - 3x^2 + 2x + 1$$

$$\Rightarrow 4x^3 - 3x^2 + 2x + 1 = \frac{8}{5} P_3(x) - 2P_2(x) + \frac{22}{5} P_1(x) \;//$$

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3.

a. $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$

Given $Q_1 = \dfrac{(x-x_0)a_1(x)}{a_0(x)}$ and $Q_2 = \dfrac{(x-x_0)^2 a_2(x)}{a_0(x)}$

$x = x_0$ is a regular singular point for the differential equation if it is analytic in Q_1 and Q_2.

And it's irregular if it is not analytic at Q_1 and Q_2.

- $2x(1-x)y'' + (5-7x)y' - 3y = 0$. To check if $x_0 = 0$ is a regular singular point, let;

$Q_1 = \dfrac{(x-x_0)(5-7x)}{2x(1-x)}$ and $Q_2 = \dfrac{(x-x_0)^2(-3)}{2x(1-x)}$

At $x_0 = 0$

$Q_1 = \dfrac{\cancel{x}(5-7x)}{2\cancel{x}(1-x)} = \dfrac{5-7x}{1-x}$ and $Q_2 = \dfrac{-3x^2}{2\cancel{x}(1-x)} = \dfrac{-3}{2-2x}$

At $x = 0$ $Q_1 = \frac{5}{1} = 5$ and $Q_2 = \frac{-3}{2}$

$\Rightarrow$ $x = 0$ is a regular singular point.

- $2x(1-x)y'' + (5-7x)y' - 3y = 0$ at $x = 0$.

By Frobenius method, let assume our solution to be;

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$\Rightarrow y'(x) = \sum_{n=0}^{\infty} a_n(n+m)x^{n+m-1} \text{ and;}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n(n+m)(n+m-1)x^{n+m-2}.$$

Substitute into the given equation, we obtain;

$$2x(1-x)y'' = 2\sum a_n(n+m)(n+m-1)x^{n+m-1} - 2\sum a_n(n+m)(n+m-1)x^{n+m}$$

$$(5-7x)y' = 5\sum a_n(n+m)x^{n+m-1} - 7\sum a_n(n+m)x^{n+m}, \text{ and}$$

$$-3y = -3\sum a_n x^{n+m}$$

Summing at terms together, we obtain;

$$\sum_{n=0}^{\infty} \left[2(n+m)(n+m-1) + 5(n+m)\right] a_n x^{n+m-1} + \left[-2(n+m)(n+m-1) - 7(n+m) - 3\right] a_n x^{n+m} = 0$$

At $n=0$

$(2m(m-1)+5m)a_0 = 0 \Rightarrow 2m^2+3m=0$ since $a_0 \neq 0$.

$\Rightarrow m=0$ or $m=-\frac{3}{2}$.

For the recurrence solution. At $m=0$

The coefficient of x^{n-1}

$\Rightarrow 2n(n-1)+5n = 2n^2+3n \Rightarrow n \geq 1$

for $n=1$

The general recurrence is gotten as

$$a_{n+1} = \frac{2(n+m)^2+5(n+m)+3}{2(n+m+1)^2+3(n+m+1)} \cdot a_n$$

At $m=0$

$$a_{n+1} = \frac{2n+3}{2n+5} a_n \quad \text{for } n \geq 0$$

For $n=0 \Rightarrow a_1 = \frac{3}{5}a_0$

For $n=1 \Rightarrow a_2 = \frac{5}{7}a_1 = \frac{5}{7}\cdot\frac{3}{5}a_0 = \frac{3}{7}a_0$

For $n=2 \Rightarrow a_3 = \frac{7}{9}a_2 = \frac{7}{9}\cdot\frac{3}{7}a_0 = \frac{3}{9}a_0$

For $n=3 \Rightarrow a_4 = \frac{9}{11}a_3 = \frac{9}{11}\cdot\frac{3}{9}a_0 = \frac{3}{11}a_0$

We can notice they are express in term of a_0.

$$y_1(x) = a_0x^0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$= a_0\left(1+\frac{3}{5} + \frac{3}{7}x^2 + \frac{3}{9}x^3 + \frac{3}{11}x^4 + \dots\right)$$

$$= 3a_0\left(1+\frac{x}{5} + \frac{x^2}{7} + \frac{x^3}{9} + \frac{x^4}{11} + \dots\right) = 3a_0\sum_{n=0}^{\infty}\frac{x^n}{2n+3}$$

$$y_1(x) = \sum_{n=0}^{\infty}\frac{3a_0}{2n+3}x^n.$$

Similarly, we can compute for $m=-\frac{3}{2}$ and obtain

$a_n = \frac{n}{n+1}a_n$ $(n \geq 0)$ and inductively all $a_n = 0$ (for $n \geq 1$)

$\Rightarrow a_0x^m = a_0x^{-3/2}$ is the only solution. $\Rightarrow y_2 = x^{-3/2}$

$$\Rightarrow y(x) = y_1 + y_2 = \sum_{n=0}^{\infty}\frac{3a_0x^n}{2n+3} + x^{-3/2} \quad //$$

4a. $(1-x^2)y'' - 2xy' + p(p+1)y = 0$

From the equation, we can see that $x = \pm 1$ is a singular point

Let $y = \sum_{n=0}^{\infty} a_n x^{m+n} \Rightarrow y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$ and

$$y'' = \sum_{n=0}^{\infty} (m+n-1)(m+n) \sum_{n=0}^{\infty} a_n x^{m+n-2}$$

Substituting into the ODE

$$(1-x^2)\sum_{n=0}^{\infty} a_n (n+m)(m+n-1) x^{m+n-2} - 2x \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$+ p(p+1) \sum_{n=0}^{\infty} a_n x^{m+r} = 0$$

Expanding similar to question (3a), we get for $n \geq 0$

The coefficient of x^{n+m} must vanish, we end up with

$$(n+2+m)(n+1+m) a_{n+2} + [p(p+1) - (n+m)(m+n+1)] a_n = 0$$

To find the indicial equation, we solve for the lowest power of x, we then get (at $n=0$)

$$m(m-1) a_0 = 0 \Rightarrow m = 0 \text{ or } m = 1.$$

Solving for the recurrence, we get

$$a_{n+2} = a_n \frac{(n+m-p)(n+m+p+1)}{(n+m+2)(n+m+1)}.$$

For $\underline{m=0}$

$$a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)} \cdot a_n$$

This will build a set of even number as the odd ones will vanish.

$$a_2 = \frac{-p(p+1)}{2} a_0$$

$$a_3 = -\frac{(p+2)(p-1)}{2 \times 3} a_1$$

$$a_4 = \frac{p(p+1)(p-2)(p+3)}{2 \times 3 \times 4} a_0$$

...

$$y_1 = x^m[a_0 + a_1 x + a_0 x^2 + a_3 x^3 + \cdots] \text{ when } m=0$$

$$= a_0\left(1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \cdots\right)$$

$$+ a_1\left(x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \cdots\right)$$

And since it terminate for odd powers

$$y = a_0\left(1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \cdots\right)$$

For $\underline{\underline{m=1}}$

$$a_{n+2} = \frac{(n+1)(n+2) - p(p+1)}{(n+3)(n+2)}a_n = \frac{(n-p)(n+p+1)}{(n+2)(n+3)}a_n$$

$$a_2 = \frac{2 - p(p+1)}{2\cdot 3}a_0 = \frac{-(p-1)(p+2)}{3!}a_0.$$

$$a_3 = \frac{-(p-2)(p+3)}{4\cdot 3}a_1 \quad , \quad a_4 = \frac{(p-1)(p+2)(p-3)(p+4)}{5!}a_0$$

We clear see that case I $(m=0)$ is similar to case II $(m=1)$.

Hence the solution of the equation is;

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

If $n = p \Rightarrow a_{n+2} = 0 \Rightarrow a_{n+4} = a_{n+6} = a_{n+8} = \cdots = 0$

If $n = p-2 \Rightarrow a_p = \dfrac{-p(p-1)}{2(2p-1)}a_p$

If $n = p-4 \Rightarrow a_{p-4} = \dfrac{-(p-2)(p-3)}{4(2p-3)}a_{p-2} = \dfrac{p(p-1)(p-2)(p-3)}{2\cdot 4(2p-1)(2p-3)}a_p$

$$\Rightarrow y(x) = a_p x^p + a_{p-2}x^{p-2} + a_{p-4}x^{p-4} + \cdots$$

$$y(x) = a_p\left[x^p - \frac{p(p-1)}{2(2p-1)}x^{p-2} + \frac{p(p-1)(p-2)(p-3)}{2\cdot 4(2p-1)(2p-3)}x^{p-4} + \cdots\right]$$

The general solution will now be

$$y(x) = \frac{(2p-1)(2p-3)\cdots 3\cdot 1}{p!}\left[x^p - \frac{p(p-1)}{2(2p-1)}x^{p-2} + \frac{p(p-1)(p-2)(p-3)}{8(2p-1)(2p-3)}x^{p-4}\right.$$

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)}x^{n-4}\right]$$

46. If $P_n(x)$ is the Legendre's polynomial of ~~orther~~ order n,

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

Recall that

$$P_n(x) = \frac{1.3.5}{n!}\left(x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-5} + \cdots\right)$$

From the generating function

$$\frac{1}{\sqrt{1+2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad (1)$$

$$\Rightarrow (1-2xt+t^2) = (1-t(xt-t))^{-\frac{1}{2}} = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots$$

Let $(1+v)^p = 1 + pv + \frac{p(p-1)}{2!}v^2 + \frac{p(p-1)(p-2)}{3!}v^3 + \cdots$

For equation (1) differentiating both side W.R.T t gives

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}\cdot(2t-2x) = n\sum_{n=0}^{\infty} P_n(x)t^{n-1}$$

$$\Rightarrow \frac{x-t}{(1-2xt+t^2)^{\frac{1}{2}}} = n\sum_{n=0}^{\infty} P_n(x)t^{n-1} \quad \left[\text{multiply through by } (1-2xt+t^2)\right]$$

$$\frac{x-t}{\sqrt{1-2xt+t^2}} = n(1-2xt+t^2)\sum_{n=0}^{\infty} P_n(x)t^{n-1}$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} P_n(x)t^n = n\sum_{n=0}^{\infty} P_n(x)t^{n-1} - 2nx\sum_{n=0}^{\infty} P_n(x)t^n + n\sum_{n=0}^{\infty} P_n(x)t^{n+1}$$

$$\Rightarrow x\sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = n\sum_{n=0}^{\infty} P_n(x)t^{n-1} - 2nx\sum_{n=0}^{\infty} P_n(x)t^n + n\sum_{n=0}^{\infty} P_n(x)t^{n+1}$$

$$\sum_{n=0}^{\infty}\left[xP_n(x) - P_{n-1}(x)\right]t^n = \sum_{n=0}^{\infty}\left[(x+1)P_{n+1}(x) - 2nxP_n(x)t\right]$$

$$\Rightarrow xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x)$$

$$(n+1)P_{n+1}(x) = (x+2nx)P_n(x) - (n-1)P_{n-1}(x) - P_{n-1}(x)$$

Dividing through by $(n+1)$

$$P_{n+1}(x) = \frac{x(2n+1)}{n+1}P_n(x) - \frac{n}{n+1}P_{n-1}(x)$$

5.a

i) $\frac{d}{dx}\left(J_K(x)\right) = \frac{1}{2}\left(J_{K-1}(x) + J_{K+1}(x)\right)$

Recall that $\frac{d}{dx}\left(x^{-K} J_K(x)\right) = x^{-K} J_{K+1}(x)$

From $\frac{d}{dx}\left(x^K J_K(x)\right) = Kx^{K-1} J_K(x) + x^K J_K'(x) = x^K J_{K-1}(x)$ ——— (1)

and $\frac{d}{dx}\left(x^{-K} J_K(x)\right) = -Kx^{-K-1} J_K(x) + x^{-K} J_K'(x) = x^{-K} J_{K+1}(x)$ ——— (2)

Divide (3) by x^K

$\frac{K}{x} J_K(x) + J_K'(x) = J_{K-1}(x)$ ——— (3)

And divide (2) by x^{-K}

$-\frac{K}{x} J_K(x) + J_K'(x) = -J_{K+1}(x)$ ——— (4)

Add equation (1) and (2)

$2J_K'(x) = J_{K-1}(x) - J_{K+1}(x)$ Divide through by 2 //.

ii) $J_{K-1}(x) + J_{K+1}(x) = \frac{2K}{x} J_K(x)$

Recall that $\sum_{n=0}^{\infty} J_n(x) t^n = e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$ (Differentiate W.R.T t)

$n\sum_{n=0}^{\infty} J_n(x) t^{n-1} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right) e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$

$\Rightarrow \sum_{n=0}^{\infty} n J_n(x) t^{n-1} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right) \sum_{n=0}^{\infty} J_n(x) t^n$ [Making t^n to be on the same power]

$\Rightarrow (n+1)\sum_{n=0}^{\infty} J_{n+1}(x) t^n = \frac{x}{2}\sum_{n=0}^{\infty} J_n(x) t^n + \frac{x}{2}\sum_{n=0}^{\infty} J_{n+2}(x) t^n$

$\sum_{n=0}^{\infty} \left[(n+1) J_{n+1}(x) = \frac{x}{2} J_n(x) + \frac{x}{2} J_{n-2}(x)\right] t^n$

$\Rightarrow (n+1) J_{n+1}(x) = \frac{x}{2} J_n(x) + \frac{x}{2} J_{n+2}(x)$ (Make $J_{n+2}(x)$ subject)

$$J_{n+2}(x) = \frac{2(n+1)}{x} J_{n+1}(x) - J_n(x)$$

□.

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b i) $\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0$, if $n \neq m$ And since $P_m(x)$ and $P_n(x)$ are Legendre's polynomials, then ;

$$(1-x^2)P_m''(x) - 2x P_m'(x) + m(m+1)P_m(x) = 0 \quad \text{——— (1)}$$

and $$(1-x^2)P_n''(x) - 2x P_n'(x) + n(n+1)P_n(x) = 0 \quad \text{——— (2)}$$

Multiply (1) by P_n and (2) by P_m

$$(1-x^2)P_m'' P_n - 2x P_m' P_n + m(m+1)P_m P_n = 0 \quad \text{——— (3)}$$

$$(1-x^2)P_n'' P_m - 2x P_n' P_m + n(n+1)P_m P_n = 0 \quad \text{——— (4)}$$

Subtract (4) from (3)

$$\Rightarrow (1-x^2)(P_m'' P_n - P_n'' P_m) - 2x(P_m' P_n - P_m P_n') + [m(m+1) - n(n+1)]P_m P_n = 0$$

$$\frac{d}{dx}\left[(1-x^2)P_n' P_m - P_m' P_n\right] = (n(n+1) - m(m+1))P_n P_m$$

Integrating both sides from -1 to 1

$$\Rightarrow (1-x^2)(P_n' P_m - P_m' P_n)\Big|_{-1}^{1} = (n(n+1) - m(m+1))\int_{-1}^{1} P_n P_m\,dx \quad n \neq m$$

$$\Rightarrow (n(n+1) - m(m+1))\int_{-1}^{1} P_n(x) P_m(x)\,dx = 0, \quad n \neq m$$

$$\Longrightarrow \int_{-1}^{1} P_n(x) P_m(x)\,dx = 0 \quad \text{if } n \neq m.$$

$\square$.

bii) $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$

From the generating function of Legendre's polynomial $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n\geq 0}^{\infty} P_n(x)t^n$

Square both side to obtain : $\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_m(x)P_n(x)t^{m+n}$

Then integrating from -1 to 1 W.R.T dx we get:

$$\int_{-1}^{1}\frac{1}{1-2xt+t^2}\,dx = \sum_{m\geq 0}^{\infty}\sum_{n\geq 0}^{\infty}\left[\int_{-1}^{1} P_m(x)P_n(x)\,dx\right]t^{m+n}$$

But $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ if $n \neq m$

$$\Longrightarrow \int_{-1}^{1}\frac{1}{1-2xt+t^2}\,dx = \sum_{n\geq 0}^{\infty}\int_{-1}^{1}\left((P_n(x))^2\,dx\right)t^{2n} \quad \text{——— (1)}$$

From the LHS $\int_{-1}^{1}\frac{1}{1-2xt+t^2}\,dx = \frac{1}{t}\left[\ln(1+t) - \ln(1-t)\right]$

$$\frac{1}{t}\left[\ln(1+t) - \ln(1-t)\right] = 2\sum_{n\geq 0}^{\infty}\frac{t^{2n}}{2n+1} \quad \text{[By series expansion]}$$

$$\Rightarrow 2\sum_{n\geq 0}^{\infty}\frac{t^{2n}}{2n+1} = \sum_{n\geq 0}^{\infty}\left[\int_{-1}^{1}(P_n(x))^2\,dx\right]t^{2n}$$

Divide through by t^{2n}

$$\frac{2}{2n+1} = \int_{-1}^{1}(P_n(x))^2\,dx \quad \square.$$